Analysis of Control Systems on Symmetric Cones

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Stability of a linear system

Is the system $\dot{x} = Ax$ asymptotically stable?

$$A = \begin{bmatrix} -10 & 1 & 5 & 1 \\ 2 & -9 & 2 & 7 \\ 1 & 0 & -41 & 0 \\ 4 & 1 & 3 & -9 \end{bmatrix}$$

- structurally dense
- no easy way to escape calculating eigenvalues, Lyapunov matrix...
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**can we do better?**

**yes** if we exploit cone structure of $A$
**Age-old problem**

**question:** When is the linear dynamical system

\[
\dot{x}(t) = Ax(t), \quad A \in \mathbb{R}^{n \times n}, \quad x(t) \in \mathbb{R}^n
\]

globally asymptotically stable? \((x(t) \to 0 \text{ as } t \to \infty \text{ for all initial conditions})\)

**answer:** solved!
Age-old problem

answer: The linear system is stable if and only if

1. all eigenvalues of the matrix $A$ have negative real part,

$$\Re(\lambda_i(A)) < 0, \quad i = 1, \ldots, n.$$
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2. given a positive definite matrix $Q = Q^T > 0$, there exists a unique matrix $P > 0$ satisfying the Lyapunov equation

\[ A^T P + PA + Q = 0. \]
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1. all eigenvalues of the matrix $A$ have negative real part,
   \[
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   \]

2. given a positive definite matrix $Q = Q^T \succ 0$, there exists a unique matrix $P \succ 0$ satisfying the Lyapunov equation
   \[
   A^T P + PA + Q = 0.
   \]

3. the following linear matrix inequality holds,
   \[
   P = P^T \succ 0, \quad A^T P + PA \prec 0.
   \]
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1. all eigenvalues of the matrix $A$ have negative real part,
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   \[ A^T P + PA + Q = 0. \]

3. the following linear matrix inequality holds,
   \[ P = P^T > 0, \quad A^T P + PA < 0. \]

4. there exists a quadratic Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}$,
   \[ V(x) = \langle x, Px \rangle \]
   which is positive definite ($V(x) > 0$ for all $x \neq 0$) and decreasing ($\dot{V} < 0$ along system trajectories).
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**answer:** The linear system is stable if and only if

\[ P = P^T \succ 0, \quad A^T P + PA \prec 0. \]
Quadratic Lyapunov function

\[ \exists P = P^T \succ 0, \quad A^T P + PA \prec 0 \]

Lyapunov’s theorem \( \Downarrow \)

\[ \dot{x} = Ax \text{ is stable} \]
**Quadratic Lyapunov function**

\[ \exists P = P^T \succ 0, \quad A^T P + PA \prec 0 \]

Lyapunov's theorem \( \downarrow \) \( \uparrow \) for all linear systems

\[ \dot{x} = Ax \text{ is stable} \]
Three cones

A proper cone is closed, convex, pointed, has nonempty interior, and closed under nonnegative scalar multiplication.

- nonnegative orthant

\[ \mathbb{R}_+^n = \{ x \in \mathbb{R}^n \mid x_i \geq 0, \text{ for all } i = 1, \ldots, n \} \]

- second order (Lorentz) cone

\[ \mathcal{L}_+^n = \{ (x_0, x_1) \in \mathbb{R} \times \mathbb{R}^{n-1} \mid \|x_1\|_2 \leq x_0 \} \]

- positive semidefinite cone

\[ \mathbb{S}_+^n = \{ X \in \mathbb{R}^{n \times n} \mid X = X^T \succeq 0 \} \]

these cones are self-dual and symmetric (cone of squares, Jordan algebra)
Cone invariance

Definition
The system $\dot{x} = Ax$ is invariant with respect to the cone $K$ if $e^{A(t)}(K) \subseteq K$.

- once the state enters $K$, it never leaves
  $x(0) \in K \Rightarrow x(t) \in K$ for all $t \geq 0$
- equivalently, $A$ is cross-positive
  $x \in K, y \in K^*$, and $\langle x, y \rangle = 0$
  $\Rightarrow \langle Ax, y \rangle \geq 0$
Linear Lyapunov function

\[ \exists p \in \mathbb{R}^n, \ p_i > 0, \ (Ap)_i < 0, \ i = 1, \ldots, n \]

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Lyapunov’s theorem \( \Downarrow \uparrow \) for \( \mathbb{R}^n_+ \)-invariant linear systems

\[ \dot{x} = Ax \text{ is stable} \]
Lorentz Lyapunov function

\[ \exists p \in \text{int } \mathcal{L}^n_+, \quad Ap \in - \text{int } \mathcal{L}^n_+ \]

Lyapunov’s theorem ⌦

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Lyapunov’s theorem \(
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\[ \dot{x} = Ax \text{ is stable} \]
General theorem

Let $L : V \to V$ be a linear operator on a Jordan algebra $V$ with corresponding symmetric cone of squares $K$, and assume that $e^L(K) \subseteq K$. The following statements are equivalent:

(a) There exists $p \succ_K 0$ such that $-L(p) \succ_K 0$

(b) There exists $z \succ_K 0$ such that $LP_z + P_zL^T$ is negative definite on $V$.

(c) The system $\dot{x}(t) = L(x)$ with initial condition $x_0 \in K$ is asymptotically stable.
**General theorem**

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(c) The system $\dot{x}(t) = L(x)$ with initial condition $x_0 \in K$ is asymptotically stable.

\[ \dot{x} = Ax \text{ is } K\text{-invariant} \]

\[ \Downarrow \]

Lyapunov function obtained by conic programming over $K$
**Simple example**

A is cross-positive (Metzler) with respect to nonnegative orthant $K = \mathbb{R}_+^n$

\[
A = \begin{bmatrix}
-10 & 1 & 5 & 1 \\
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\end{bmatrix} $$

- linear Lyapunov function $V = \langle p, x \rangle$ suffices:

$$ p = \begin{bmatrix}
1.4392 \\
2.7788 \\
0.22079 \\
1.9704
\end{bmatrix} \succcurlyeq_{\mathbb{R}^n_+} 0 \quad Ap = \begin{bmatrix}
-8.5383 \\
-7.8967 \\
-7.6133 \\
-8.536
\end{bmatrix} \succcurlyeq_{\mathbb{R}^n_+} 0 $$
Simple example

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-7.8967 \\
-7.6133 \\
-8.536
\end{bmatrix} \prec_0 \mathbb{R}_+^n
\]

- quadratic representation: there exists $z \in \mathbb{R}_+^n$ such that $p = z \circ z$

\[
V(x) = \langle x, P_z x \rangle, \quad P_z = \text{diag}(z^2), \quad z = \begin{bmatrix}
\sqrt{1.4392} \\
\sqrt{2.7788} \\
\sqrt{0.22079} \\
\sqrt{1.9704}
\end{bmatrix}
\]
Transportation network example

- Directed transportation network $x_1, \ldots, x_4$ (Rantzer, 2012), augmented with a catch-all buffer $x_0$.

\[
\begin{bmatrix}
\dot{x}_0 \\
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3 \\
\dot{x}_4 
\end{bmatrix} =
\begin{bmatrix}
l_{00} & l_{01} & l_{02} & l_{03} & l_{04} \\
0 & -1 - l_{31} & l_{12} & 0 & 0 \\
0 & 0 & -l_{12} - l_{32} & l_{23} & 0 \\
l_{31} & l_{32} & -l_{23} - l_{43} & l_{34} & 0 \\
0 & 0 & 0 & l_{43} & -4 - l_{34}
\end{bmatrix}
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x_0 \\
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\]
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- Metzler substructure

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\dot{x}_0 \\
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\dot{x}_2 \\
\dot{x}_3 \\
\dot{x}_4
\end{bmatrix} =
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\ell_{00} & \ell_{01} & \ell_{02} & \ell_{03} & \ell_{04} \\
0 & -1 - \ell_{31} & \ell_{12} & 0 & 0 \\
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Transportation network example

- Directed transportation network $x_1, \ldots, x_4$ (Rantzer, 2012), augmented with a catch-all buffer $x_0$.
- Metzler substructure
- $\ell_{00}, \ldots, \ell_{04}$ have no definite sign

$$
\begin{bmatrix}
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\end{bmatrix}
\begin{bmatrix}
x_0 \\
x_1 \\
x_2 \\
x_3 \\
x_4 \\
\end{bmatrix}
$$
Lorentz cone-invariant dynamics

Figure 1: Embedded focus along $x_0$-axis

A dynamics matrix $A$ is $\mathcal{L}_+^n$-invariant if and only if there exists

$$\xi \in \mathbb{R}, \quad A^T J_n + J_n A - \xi J_n \succeq 0. \quad (*)$$

Provided this condition holds, $A$ is (Hurwitz) stable if and only if there exists $p \succ \mathcal{L}_+^n 0$ with $Ap \prec \mathcal{L}_+^n 0$. 


## Technical summary

<table>
<thead>
<tr>
<th>Algebra:</th>
<th>Real</th>
<th>Lorentz</th>
<th>Symmetric</th>
</tr>
</thead>
<tbody>
<tr>
<td>$V$</td>
<td>$\mathbb{R}^n$</td>
<td>$\mathbb{R}^n$</td>
<td>$\mathbb{S}^n$</td>
</tr>
<tr>
<td>$K$</td>
<td>$\mathbb{R}^+_n$</td>
<td>$\mathbb{L}^+_n$</td>
<td>$\mathbb{S}^+_n$</td>
</tr>
<tr>
<td>$\langle x, y \rangle$</td>
<td>$x^T y$</td>
<td>$x^T y$</td>
<td>$\text{Tr}(XY^T)$</td>
</tr>
<tr>
<td>$x \circ y$</td>
<td>$x_i y_i$</td>
<td>$z^T z - \frac{z^T J_n z}{2} J_n$</td>
<td>$\frac{1}{2}(XY + YX)$</td>
</tr>
<tr>
<td>$P_z, z \in \text{int } K$</td>
<td>$\text{diag}(z)^2$</td>
<td>$x^T \left( z z^T - \frac{z^T J_n z}{2} J_n \right) x$</td>
<td>$X \leftrightarrow ZXZ$</td>
</tr>
<tr>
<td>$V(x) = \langle x, P_z(x) \rangle$</td>
<td>$x^T \text{diag}(z)^2 x$</td>
<td>$x^T \left( z z^T - \frac{z^T J_n z}{2} J_n \right) x$</td>
<td>$|XZ|_F^2$</td>
</tr>
</tbody>
</table>

Free variables in $V(x)$
- $x \mapsto Ax$
- $A$ is Metzler
- $A$ satisfies $(\ast)$
- $\|\langle Ap \rangle_1\|_2 < \langle Ap \rangle_0$ 

Dynamics $L$
- $L$ is cross-positive
- $-L(p) \succ_K 0$

Stability verification
- $\text{LP}$
- $\text{SOCP}$
- $\text{SDP}$

### Table 1: Summary of dynamics preserving a cone
Questions

Is (say) $H_\infty$ control synthesis possible via

- LP?
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- SOCP? (conjecture: yes)

Are these three cones the end of the story?

- yes (kind of)
Symmetric cone categorization

If $K$ is a (finite dimensional) symmetric cone, then it is a cartesian product

$$K = K_1 \times K_2 \times \cdots \times K_N,$$

where each $K_i$ is one of (e.g., Faraut 1994)

- $n \times n$ self-adjoint positive semidefinite matrices with real, complex, or quaternion entries
- $3 \times 3$ self-adjoint positive semidefinite matrices with octonion entries (Albert algebra), and
- Lorentz cone
Contributions

- analysis idea comes from the cone inclusion
  
  \[
  \text{nonnegative orthant} \subseteq \text{second-order cone} \subseteq \text{semidefinite cone}
  \]
  
  \[
  LP \subseteq \text{SOCP} \subseteq SDP
  \]
  
  easy $\rightarrow$ harder $\rightarrow$ hardest

- characterized new class of linear systems that admit SOCP-based analysis without any loss

- unified existing analysis frameworks

- algebraic connections with a mature theory (Jordan algebras)
Thanks!


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