Analysis of Control Systems on Symmetric Cones

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Stability of a linear system

Is the system $\dot{x} = Ax$ asymptotically stable?

$$A = \begin{bmatrix} -10 & 1 & 5 & 1 \\ 2 & -9 & 2 & 7 \\ 1 & 0 & -41 & 0 \\ 4 & 1 & 3 & -9 \end{bmatrix}$$

- structurally dense
- no easy way to escape calculating eigenvalues, Lyapunov matrix...

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can we do better?

yes if we exploit cone structure of A

Age-old problem

question: When is the linear dynamical system

$$\dot{x}(t) = Ax(t), \quad A \in \mathbf{R}^{n \times n}, \quad x(t) \in \mathbf{R}^n$$

globally asymptotically stable? ($x(t) \to 0$ as $t \to \infty$ for all initial conditions)

answer: solved!

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4. there exists a quadratic Lyapunov function $V : \mathbf{R}^n \to \mathbf{R}$,

$$V(x) = \langle x, Px \rangle$$

which is positive definite ($V(x) > 0$ for all $x \neq 0$) and decreasing ($\dot{V} < 0$ along system trajectories).

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Quadratic Lyapunov function

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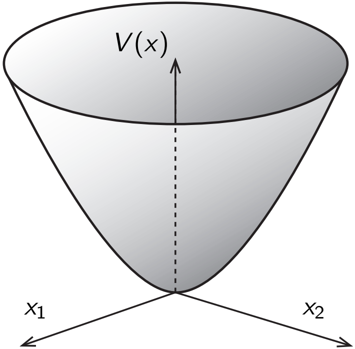
Lyapunov's theorem $\Downarrow$

$$\dot{x} = Ax \text{ is stable}$$



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Lyapunov's theorem $\Downarrow$ $\Uparrow$ for all linear systems

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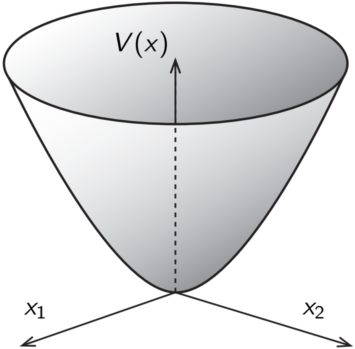


Three cones

A proper cone is closed, convex, pointed, has nonempty interior, and closed under nonnegative scalar multiplication.

- nonnegative orthant

$$\mathbf{R}^n_+ = \{x \in \mathbf{R}^n \mid x_i \geq 0, \text{ for all } i = 1, \ldots, n\}$$

- second order (Lorentz) cone

$$\mathcal{L}^n_+ = \{(x_0, x_1) \in \mathbf{R} \times \mathbf{R}^{n-1} \mid \|x_1\|_2 \leq x_0\}$$

- positive semidefinite cone

$$\mathbf{S}^n_+ = \{X \in \mathbf{R}^{n \times n} \mid X = X^T \succeq 0\}$$

these cones are self-dual and symmetric (cone of squares, Jordan algebra)

Cone invariance

Definition

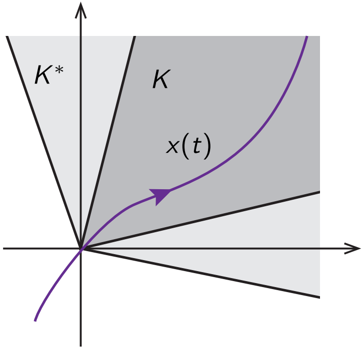
The system $\dot{x} = Ax$ is invariant with respect to the cone K if $e^{A}(K) \subseteq K$.

- once the state enters K, it never leaves

$$x(0) \in K \Rightarrow x(t) \in K \text{ for all } t \geq 0$$

- equivalently, A is cross-positive

$$x \in K, y \in K^*, \text{ and } \langle x, y \rangle = 0 \\ \Rightarrow \langle Ax, y \rangle \geq 0$$



Linear Lyapunov function

$$\exists p \in \mathbf{R}^n, \quad p_i > 0, \quad (Ap)_i < 0, \quad i = 1, \ldots, n$$

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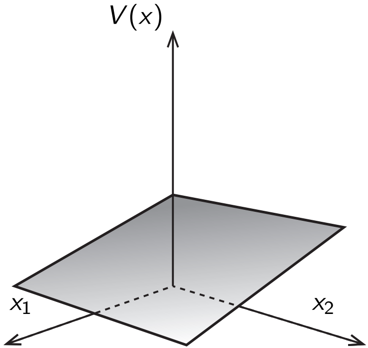

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Lyapunov's theorem $\Downarrow$ $\Uparrow$ for $\mathbf{R}_+^n$-invariant linear systems

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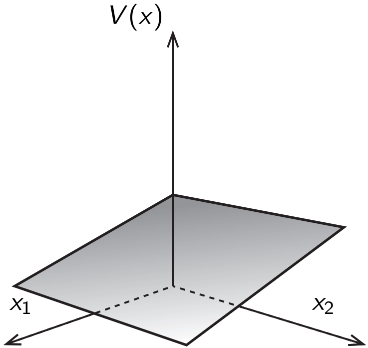

Lorentz Lyapunov function

$$\exists p \in \textbf{int}\,\mathcal{L}_+^n, \quad Ap \in -\,\textbf{int}\,\mathcal{L}_+^n$$

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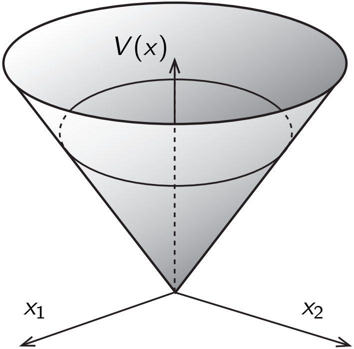

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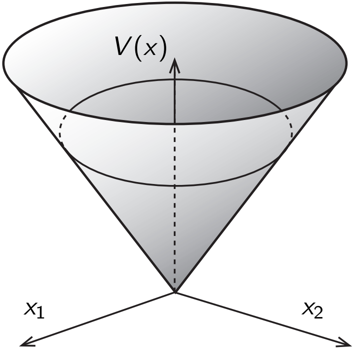

General theorem

Let $L : V \to V$ be a linear operator on a Jordan algebra V with corresponding symmetric cone of squares K, and assume that $e^L(K) \subseteq K$. The following statements are equivalent:

(a) There exists $p \succ_K 0$ such that $-L(p) \succ_K 0$

(b) There exists $z \succ_K 0$ such that $LP_z + P_z L^T$ is negative definite on V.

(c) The system $\dot{x}(t) = L(x)$ with initial condition $x_0 \in K$ is asymptotically stable.

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$\dot{x} = Ax$ is K-invariant

$\Downarrow$

Lyapunov function obtained by conic programming over K

Simple example

A is cross-positive (Metzler) with respect to nonnegative orthant $K = \mathbf{R}^n_+$

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- linear Lyapunov function $V = \langle p, x \rangle$ suffices:

$$p = \begin{bmatrix} 1.4392 \\ 2.7788 \\ 0.22079 \\ 1.9704 \end{bmatrix} \succ_{\mathbf{R}^n_+} 0 \quad Ap = \begin{bmatrix} -8.5383 \\ -7.8967 \\ -7.6133 \\ -8.536 \end{bmatrix} \prec_{\mathbf{R}^n_+} 0$$

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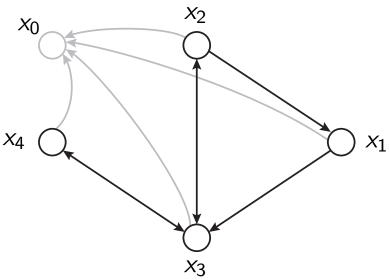
- quadratic representation: there exists $z \in \mathbf{R}^n_+$ such that $p = z \circ z$

$$V(x) = \langle x, P_z x \rangle, \quad P_z = \mathbf{diag}(z^2), \quad z = \begin{bmatrix} \sqrt{1.4392} \\ \sqrt{2.7788} \\ \sqrt{0.22079} \\ \sqrt{1.9704} \end{bmatrix}$$

Transportation network example



- Directed transportation network $x_1, \dots, x_4$ (Rantzer, 2012), augmented with a catch-all buffer x_0.

$$\begin{bmatrix} \dot{x}_0 \\ \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} \ell_{00} & \ell_{01} & \ell_{02} & \ell_{03} & \ell_{04} \\ 0 & -1-\ell_{31} & \ell_{12} & 0 & 0 \\ 0 & 0 & -\ell_{12}-\ell_{32} & \ell_{23} & 0 \\ 0 & \ell_{31} & \ell_{32} & -\ell_{23}-\ell_{43} & \ell_{34} \\ 0 & 0 & 0 & \ell_{43} & -4-\ell_{34} \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

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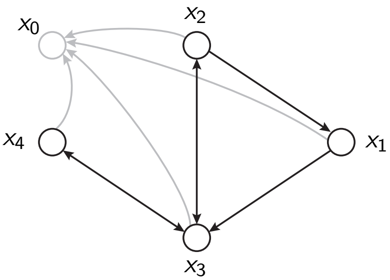

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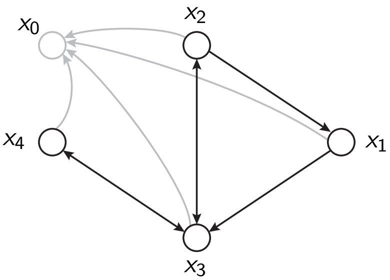

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- Metzler substructure
- $\ell_{00}, \dots, \ell_{04}$ have no definite sign

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Lorentz cone-invariant dynamics

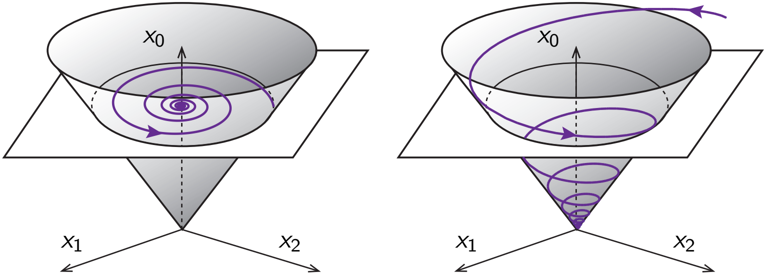

Figure 1: Embedded focus along x_0-axis

A dynamics matrix A is $\mathcal{L}_+^n$-invariant if and only if there exists

$$\xi \in \mathbf{R}, \quad A^T J_n + J_n A - \xi J_n \succeq 0. \qquad (*)$$

Provided this condition holds, A is (Hurwitz) stable if and only if there exists $p \succ_{\mathcal{L}_+^n} 0$ with $Ap \prec_{\mathcal{L}_+^n} 0$.

Technical summary

Algebra:	Real	Lorentz	Symmetric
V	$\mathbf{R}^n$	$\mathbf{R}^n$	$\mathbf{S}^n$
K	$\mathbf{R}^n_+$	$\mathcal{L}^n_+$	$\mathbf{S}^n_+$
$\langle x, y \rangle$	$x^T y$	$x^T y$	$\mathbf{Tr}(XY^T)$
$x \circ y$	$x_i y_i$	$(x^T y, x_0 y_1 + y_0 x_1)$	$\frac{1}{2}(XY + YX)$
$P_z,\ z \in \mathbf{int}\, K$	$\mathbf{diag}(z)^2$	$zz^T - \frac{z^T J_n z}{2} J_n$	$X \mapsto ZXZ$
$V(x) = \langle x, P_z(x) \rangle$	$x^T \mathbf{diag}(z)^2 x$	$x^T \left(zz^T - \frac{z^T J_n z}{2} J_n \right) x$	$\lVert XZ \rVert_F^2$
Free variables in $V(x)$	n	n	$n(n+1)/2$
dynamics L	$x \mapsto Ax$	$x \mapsto Ax$	$X \mapsto AX + XA^T$
L is cross-positive	A is Metzler	A satisfies $(*)$	by construction
$-L(p) \succ_K 0$	$(Ap)_i < 0$	$\lVert (Ap)_1 \rVert_2 < (-Ap)_0$	$AP + PA^T \prec 0$
Stability verification	LP	SOCP	SDP

Table 1: Summary of dynamics preserving a cone

Questions

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- yes (kind of)

Symmetric cone categorization

If K is a (finite dimensional) symmetric cone, then it is a cartesian product

$$K = K_1 \times K_2 \times \cdots \times K_N,$$

where each K_i is one of (*e.g.*, Faraut 1994)

- $n \times n$ self-adjoint positive semidefinite matrices with real, complex, or quaternion entries
- 3×3 self-adjoint positive semidefinite matrices with octonion entries (Albert algebra), and
- Lorentz cone

Contributions

- analysis idea comes from the cone inclusion

$$\text{nonnegative orthant} \subseteq \text{second-order cone} \subseteq \text{semidefinite cone}$$

$$\text{LP} \subseteq \text{SOCP} \subseteq \text{SDP}$$

$$\text{easy} \rightarrow \text{harder} \rightarrow \text{hardest}$$

- characterized new class of linear systems that admit SOCP-based analysis without any loss
- unified existing analysis frameworks
- algebraic connections with a mature theory (Jordan algebras)

Thanks!

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`www.cds.caltech.edu/~ipapusha`

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