Lecture 1. Introduction

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CDS270–2: Mathematical Methods in Control and System Engineering

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Logistics

history.

- this is a *new course* taught at Caltech CDS for the first time
- one of a kind, unorthodox, groundbreaking, innovative, historic, our best idea yet
- there may be bugs (in homeworks, lectures, backhanded compliments)

bug report policy. please do!

Logistics

lectures.

- frequency: 3.3×10^{-6}Hz $\quad (2 \times 55$min each week)
- mostly by: Ivan Papusha
- time: please fill out online survey by **Tue, Apr 1**
- first "real" lecture: **next week** after we determine a time
- place: please check back at the website
- website:

  `http://www.cds.caltech.edu/~ipapusha/cds270/`

homework policy. please do!

Homeworks

- weekly readings
- weekly homeworks with 1-2 problems
- grading: $\checkmark-$ / $\checkmark$ / $\checkmark+$
- breakdown: 50% homework / 50% participation
- **hw1** is assigned and is due **next week**
- hw2+ will be “choose your own adventure”:

do an assigned problem

or

pick and do a problem from a catalog

- catalog updated frequently

Themes

- Dynamical Systems
- Lyapunov theory
- Convex optimization
- Linear Matrix Inequalities

Dynamical systems

We generally think of dynamical systems as *initial value problems* or *ordinary differential equations* on a *state space*,

$$\dot{x}(t) = f(x(t)), \quad t \geq 0$$
$$x(0) = x_0$$

- $x(t) \in \mathbf{R}^n$, for all times $t \geq 0$, where $\mathbf{R}^n$ is the state space
- $x_0 \in \mathbf{R}^n$ is the initial condition

Linear dynamical systems

This class will amost exclusively focus on *linear* and *time invariant* dynamical systems

$$\begin{aligned} \dot{x}(t) &= Ax(t), \quad t \geq 0 \\ x(0) &= x_0 \end{aligned}$$

- dynamics determined by matrix $A \in \mathbf{R}^{n \times n}$, and the initial condition $x_0 \in \mathbf{R}^n$
- solution given by the *matrix exponential*

  $$x(t) = e^{At} x_0,$$

  where

  $$e^M = I + M + \frac{1}{2!} M^2 + \cdots$$

Adding a control variable

fact. many LTI systems can be written in (A, B, C, D) form

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \\ x(0) = x_0 \end{cases}$$

- *input*: $u(t) \in \mathbf{R}^m$
- *output*: $y(t) \in \mathbf{R}^p$
- *state*: $x(t) \in \mathbf{R}^n$
- assuming $u(t)$ is causal with $u(t) = 0$ for $t < 0$, the convolution equation for output is

$$y(t) = Ce^{At}x_0 + \int_{0^-}^{t} Ce^{A(t-\tau)}Bu(\tau)\,d\tau + Du(t)$$

Lyapunov theory

example result. The autonomous system

$$\dot{x}(t) = Ax(t), \quad x(t) \in \mathbf{R}^n,$$

is asymptotically stable if and only if

- all eigenvalues of $A \in \mathbf{R}^{n \times n}$ have negative real part
- there exists a quadratic Lyapunov function

$$V(x) = x^T P x, \quad P = P^T \succ 0,$$
$$\dot{V}(x) = x^T (A^T P + PA) x < 0 \quad \text{for all } x \neq 0$$

- the system of **linear matrix inequalities**

$$P \succ 0, \quad A^T P + PA \prec 0$$

is feasible for some $P = P^T \in \mathbf{R}^{n \times n}$

Robust stability

Consider the uncertain system (not LTI)

$$\dot{x}(t) \in \Omega x(t), \quad x(t) \in \mathbf{R}^n,$$

where Ω is a subset of $\mathbf{R}^{n \times n}$

- this is a *differential inclusion* (*cf.* differential equation)
- example sets

$$\begin{aligned}
\Omega &= \{A\}, \\
\Omega &= \{A_1, A_2, \ldots, A_L\}, \\
\Omega &= \{A + B\Delta C \mid \lambda_{\max}(\Delta^T \Delta) \leq 1\}
\end{aligned}$$

Polytopic LDI

The linear differential inclusion

$$\dot{x}(t) \in \Omega x(t), \quad \Omega = \mathrm{conv}\{A_1, A_2, \ldots, A_L\}$$

has all trajectories converge to zero as $t \to \infty$ if there exists a joint Lyapunov function $V(x) = x^T P x$,

$$P \succ 0, \quad A_i^T P + P A_i \prec 0, \quad i = 1, \ldots, L$$

- a system of linear matrix inequalities in $P = P^T \in \mathbf{R}^{n \times n}$
- no closed form solution
- algorithms based on **linear algebra** and **convex optimization** can be used to find P

Convex optimization

Many problems in control and dynamical systems reduce to an *optimization* problem

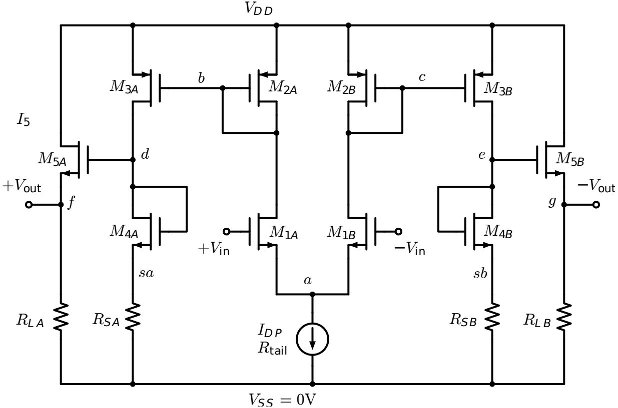
$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & x \in \mathcal{C} \end{array}$$

over the variable $x \in \mathbf{R}^n$

- the problem is convex if $f_0 : \mathbf{R}^n \to \mathbf{R}$ is a *convex function* and $\mathcal{C}$ is a *convex set*.
- optimal value $x^\star$ satisfies $f_0(x^\star) \leq f(x)$ for all $x \in \mathcal{C}$
- feasibility problem if $f_0(x) = 0$
- convex optimization is a rich field of study with computational teeth

Application: circuit sizing

design variables: $W_1/L_1, \dots, W_5/L_5$



minimize	quiescent power
subject to	$M_1, \dots, M_5$ in saturation
	$CMRR \geq 80\text{dB}$

Linear matrix inequalities

For linear dynamical systems, many specifications and robust analysis/synthesis can be expressed as a *semidefinite program* (SDP)

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & F_0 + x_1 F_1 + \cdots x_n F_n \succeq 0 \end{array}$$

with variable $x \in \mathbf{R}^n$ and parameters $c \in \mathbf{R}^n$, $F_i = F_i^T \in \mathbf{R}^{n \times n}$

- for a symmetric matrix $M = M^T \in \mathbf{R}^{n \times n}$,

$$M \succeq 0 \text{ means } x^T M x \geq 0 \text{ for all } x \in \mathbf{R}^n$$

- challenge is to write down the SDP
- in theory: if we can write a problem as an SDP, it can be solved by an algorithm
- in practice: these days, only if F_i are 50×50 or so