Lecture 3. Lyapunov Theory

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CDS270–2: Mathematical Methods in Control and System Engineering

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Logistics

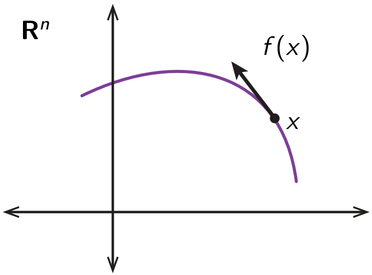
- hw2 due this **Wed, Apr 15**
  - do an easy problem or CYOA
- hw1 solutions posted online
- start reading: lmibook, Ch 1–2
  - the book is dense, but *extremely* good
  - free online, written in 1994—even more timely now than ever
  - less important on a first reading: §2.3–2.4 (algorithms)
  - very important: §2.6.3 ($\mathcal{S}$-procedure), §2.7.2–3 (KYP)

Dynamical systems

A dynamical system concerns quantities that evolve in time, *e.g.*,

$$\dot{x}(t) = f(x(t)), \quad x(0) = x_0.$$

Here $x(t) \in \mathbf{R}^n$ is a state variable, and $f : \mathbf{R}^n \to \mathbf{R}^n$ is the (infinitesimal) direction of evolution.



Solutions to ODEs

Often do not want (or care) to compute $x(t)$ directly in closed form

$$\dot{x} = f(x), \quad x(0) = x_0$$

- if $f(x) = Ax$, then $x(t) = e^{At}x_0$
- **fact.** if f is Lipschitz in a neighborhood of x_0, then the following algorithm converges to a unique solution (locally)

$$\begin{aligned} x^{(0)}(t) &:= x_0 \\ x^{(k+1)}(t) &:= x_0 + \int_0^t f(x^{(k)}(\tau))\, d\tau, \quad k = 0, 1, 2, \ldots \end{aligned}$$

- time integration methods (Euler, RK, symplectic, ...)

Conserved quantities

Let $V : \mathbf{R}^n \to \mathbf{R}$ be a real-valued function on a state space. We say that V is a *conserved quantity* if it is constant,

$$\dot{V}(x) = \nabla V(x)^T f(x) = 0,$$

along trajectories of $\dot{x} = f(x)$

- $\dot{V}$ is a *Lie* derivative along vector field f
- trajectories stay in level sets of V,

$$\{z \in \mathbf{R}^n \mid V(z) = \alpha\}$$

proof. if $V(x(0)) = \alpha$, then

$$V(x(t)) = \alpha + \int_0^t \underbrace{\dot{V}(x(\tau))}_{=0} \, d\tau = \alpha$$

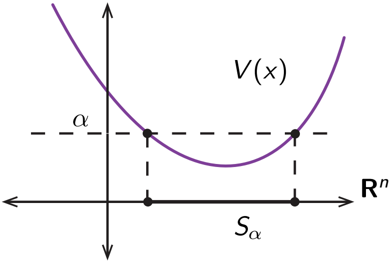
for all $t \geq 0$

Sublevel sets

The α-sublevel set of a function $V : \mathbf{R}^n \to \mathbf{R}$ is

$$S_\alpha = \{z \in \mathbf{R}^n \mid V(z) \leq \alpha\}$$

- S_α can be unbounded
- if V is convex, then so is S_α



Dissipated quantities

Let $V : \mathbf{R}^n \to \mathbf{R}$ be a real-valued function on a state space. We say that V is a *dissipated quantity* if it is nonincreasing,

$$\dot{V}(x) = \nabla V(x)^T f(x) \leq 0,$$

along trajectories of $\dot{x} = f(x)$

- $-\dot{V}$ is the *dissipation rate*
- trajectories stay in sublevel sets of V,

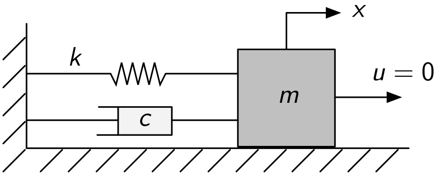
$$S_\alpha = \{z \in \mathbf{R}^n \mid V(z) \leq \alpha\}$$

proof. if $V(x(0)) \leq \alpha$, then

$$V(x(t)) = V(x(0)) + \int_0^t \underbrace{\dot{V}(x(\tau))}_{\leq 0} \, d\tau \leq \alpha$$

for all $t \geq 0$

Example: spring-mass-dashpot



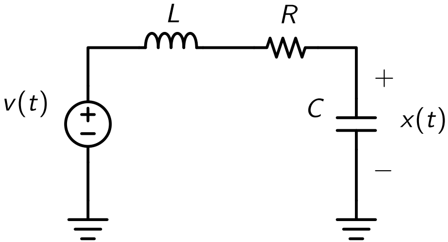
$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = 0 \quad \Longleftrightarrow \quad \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

- total energy: $V(x_1, x_2) = \frac{1}{2}kx_1^2 + \frac{1}{2}mx_2^2$
- energy derivative:

$$\dot{V}(x_1, x_2) = \begin{bmatrix} kx_1 \\ mx_2 \end{bmatrix}^T \left(\begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) = -cx_2^2$$

- V is conserved if $c = 0$, dissipated if $c > 0$

Example: capacitor-inductor-resistor



$$LC\ddot{x}(t) + RC\dot{x}(t) + x(t) = v(t)$$

(compare to)

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = u(t)$$

- large inductors are like heavy masses
- small capacitors are like stiff springs
- resistors dissipate energy

Positive definite functions

A function $V : \mathbf{R}^n \to \mathbf{R}$ is *positive definite* if

- $V(x) \geq 0$ for all x
- $V(x) = 0$ if and only if $x = 0$
- all sublevel sets of V are bounded

example. the function $V(x) = x^T P x$ is positive definite $\iff P \succ 0$.

Lyapunov stability theorem

Suppose there is a function $V : \mathbf{R}^n \to \mathbf{R}$ such that

- *Generalized energy:* V is positive definite
- *Strict dissipation:* $\dot{V}(x) < 0$ for all $x \neq 0$ and $\dot{V}(0) = 0$

then every trajectory of $\dot{x} = f(x)$ converges to zero as $t \to \infty$.

proof. Suppose $x(t) \not\to 0$. Since V is a dissipated, nonnegative quantity, $V \geq 0$ and $\dot{V} \leq 0$ together mean that $V \to c_1 > 0$. In particular, $c_1 \leq V(x(t)) \leq V(x(0)) = c_2$ for all $t \geq 0$. Take

$$C = \{z \in \mathbf{R}^n \mid 0 < c_1 \leq V(z) \leq c_2\}.$$

Since $C \subset S_{c_2}$ is compact and V is strictly dissipated, we have $\sup_{z \in C} \dot{V}(z) = -\gamma < 0$. But the energy at time t,

$$V(x(t)) = V(x(0)) + \int_0^t \underbrace{\dot{V}(x(\tau))}_{\leq -\gamma} d\tau \leq c_2 - \gamma t,$$

is negative for large t, a contradiction.

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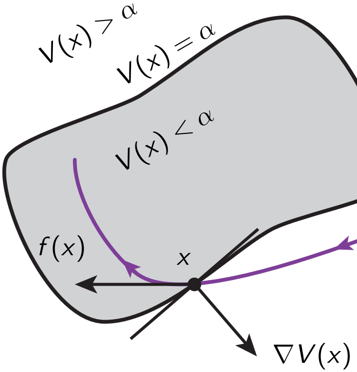
Graphical interpretation

If the sublevel sets of V are bounded and V is

- conserved: $\nabla V(x)^T f(x) = 0 \implies x(t)$ moves along level set
- dissipated: $\nabla V(x)^T f(x) \leq 0 \implies x(t)$ cannot escape sublevel set
- strictly dissipated: $\nabla V(x)^T f(x) < 0 \implies x(t)$ enters sublevel set



Other Lyapunov-like results

non strict dissipation. if $\dot{V}(x) \leq 0$, then trajectories can hide in the zero-dissipation set

$$\{z \in \mathbf{R}^n \mid \dot{V}(z) = 0\},$$

but if the only solution to $\dot{x} = f(x)$, $\dot{V}(x) = 0$, is $x(t) \equiv 0$ for all t, then $x(t) \to 0$ (LaSalle)

decay rate. if the dissipation rate is $-\dot{V} \geq 2\alpha V$, then trajectories of $\dot{x} = f(x)$ decay exponentially with rate at least α (ex. 3),

$$\lim_{t\to\infty} e^{\alpha t}\|x(t)\|_2 = 0$$

region of attraction. define $\mathcal{R} = \{x_0 \in \mathbf{R}^n \mid \lim_{t\to\infty} x(t) = 0\}$. if

$$S_\alpha = \{z \in \mathbf{R}^n \mid V(z) \leq \alpha\} \subseteq \mathcal{D} := \{z \in \mathbf{R}^n \mid \dot{V}(z) < 0\} \cup \{0\},$$

then $S_\alpha \subseteq \mathcal{R}$, *i.e.*, S_α is an inner approximation of $\mathcal{R}$.

Central idea

If we can find an energy-like (Lyapunov) function $V : \mathbf{R}^n \rightarrow \mathbf{R}$, that satisfies certain dissipation conditions, **then** we can conclude something about the trajectories of the system, *e.g.*

- stability
- robustness wrt. parameter perturbations
- decay rate
- input and output energy bounds
- bounds on peak, overshoot
- regions of attraction...

Where to get Lyapunov functions:

- physical insight
- Lyapunov function from system linearization
- more sophisticated methods (sum of squares...)

Example: region of attraction

Nonlinear system (Van der Pol oscillator with time reversed)

$$\begin{cases} \dot{x}_1 = -x_2 \\ \dot{x}_2 = x_1 + (x_1^2 - 1)x_2 \end{cases}$$

- linearization about equilibrium $(0, 0)$ is stable

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \lambda_i = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} j$$

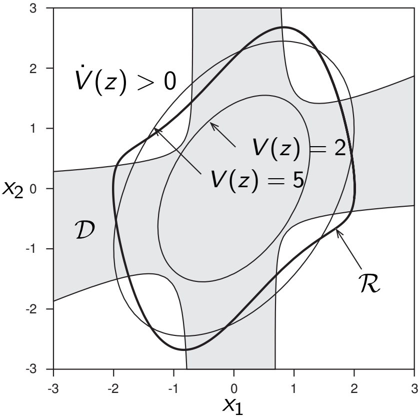
- suggests quadratic Lyapunov function $V(z) = z^T P z$, *e.g.*,

$$V(z) = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}^T \underbrace{\begin{bmatrix} 1.5 & -0.5 \\ -0.5 & 1 \end{bmatrix}}_{P} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}, \quad A^T P + PA = -I \prec 0$$

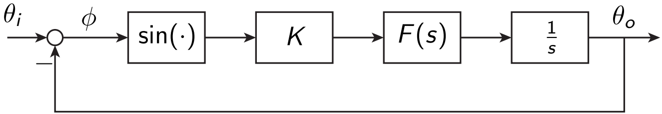
Example: region of attraction

$$\dot{V}(z) = 2z^T P \dot{z} = 2 \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}^T \begin{bmatrix} 1.5 & -0.5 \\ -0.5 & 1 \end{bmatrix} \begin{bmatrix} -z_2 \\ z_1 + (z_1^2 - 1) z_2 \end{bmatrix}$$
$$= -(z_1^2 + z_2^2) - (z_1^3 z_2 - 2 z_1^2 z_2^2)$$

- strict dissipation set (shaded):
  $\mathcal{D} = \{z \mid \dot{V}(z) < 0\} \cup \{0\}$
- largest ellipsoidal sublevel in $\mathcal{D}$:
  $S_\alpha = \{z \mid V(z) \leq 2.25\}$
- true region of attraction $\mathcal{R}$ enclosed by limit cycle



Engineering example: phase locked loop[1]



$$\ddot{\phi} + K\frac{\tau_2}{\tau_1}\cos(\phi)\dot{\phi} + \frac{K}{\tau_1}\sin(\phi) = \ddot{\theta}_i, \quad F(s) = \frac{1+\tau_2 s}{\tau_1 s}$$

- $K, \tau_1, \tau_2 > 0$ are design parameters
- in state space $x_1 = \phi$, $x_2 = -\dot{\phi}$, and with $\theta_i = 0$

$$\begin{cases} \dot{x}_1 = -x_2 \\ \dot{x}_2 = \dfrac{K}{\tau_1}\sin(x_1) - K\dfrac{\tau_2}{\tau_1}\cos(x_1)x_2 \end{cases}$$

- reverse time Van der Pol with $\sin(x_1) \approx x_1$ and $\cos(x_2) \approx (1 - x_2^2/2)$

[1] T-C Wang, S. Lall, T-Y Chiou. Polynomial Method for PLL Controller Optimization. *Sensors* 11(7):6575–6592, 2011.

Summary: estimating the region of attraction

strict dissipation set: $\mathcal{D} = \{z \mid \dot{V}(z) < 0\} \cup \{0\}$
bounded energy sublevels: $S_\alpha = \{z \mid V(z) \leq \alpha\}$
region of attraction: $\mathcal{R} = \{x_0 \mid \lim_{t\to\infty} x(t) = 0\}$

- Trajectories starting at a point $x_0 \in \mathcal{D}$ with initial energy $V(x_0) = \alpha$ must stay within S_α.
- If S_α contains a point outside $\mathcal{D}$, a trajectory through that point can gain energy and escape S_α.
- If S_α is entirely within $\mathcal{D}$, no trajectory can escape S_α.

therefore $S_\alpha \subseteq \mathcal{D}$ implies $S_\alpha \subseteq \mathcal{R}$.

nonstrict dissipation regions can be used to compute *invariant sets*

Invariant ellipsoids

- for quadratic Lyapunov functions $V(z) = z^T P z$, the energy sublevels are ellipsoids,
$$S_\alpha = \{z \in \mathbf{R}^n \mid z^T P z \leq \alpha\}$$
- for (marginally) stable linear state space systems, nonstrict dissipation sets are all of $\mathbf{R}^n$
$$\mathcal{D} = \{z \in \mathbf{R}^n \mid \dot{V}(z) = z^T(A^T P + PA)z \leq 0\},$$
$$\text{hence } S_\alpha \subseteq \mathcal{D} = \mathbf{R}^n$$
- thus linear state space systems are either globally (marginally) stable, or not globally (marginally) stable

much more interesting in the study of state–output and input–output properties of LDIs

Lyapunov stability theorem for linear systems

For the state space system $\dot{x} = Ax$, $V(z) = z^T P z$, and

$$\dot{V}(z) = z^T(A^T P + PA)z = -z^T Q z,$$

if $P \succ 0$, $Q \succ 0$, then $x(t) \to 0$.

- **converse.** if $\dot{x} = Ax$ is stable, then there exists $P \succ 0$ and $Q \succ 0$ to prove it. (Lyapunov is exact for linear systems.)
- typically fix $Q = Q^T \succ 0$ and solve Lyapunov equation

$$A^T P + PA + Q = 0$$

- solution given by the Gramian

$$P = \int_0^\infty e^{A^T \tau} Q e^{A\tau}\, d\tau$$

Lyapunov equation

The Lyapunov equation $A^T P + PA + Q = 0$ really is a set of linear equations, *e.g.*, for $n = 2$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^T \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} + \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = - \begin{bmatrix} q_1 & q_2 \\ q_2 & q_3 \end{bmatrix}$$

can be rewritten as

$$\begin{bmatrix} 2a_{11} & 2a_{21} & 0 \\ a_{12} & a_{11} + a_{22} & a_{21} \\ 0 & 2a_{12} & 2a_{22} \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} -q_1 \\ -q_2 \\ -q_3 \end{bmatrix}$$

in matlab: `P = lyap(A',Q);  % note the transpose!`

Observability

useful fact. (PBH test) The pair (A, C) is observable if and only if there exists no $x \neq 0$ such that

$$Ax = \lambda x, \quad Cx = 0.$$

- if such $x \neq 0$ exists, then

$$\begin{aligned} Cx &= 0 \\ CAx &= \lambda Cx = 0 \\ CA^2x &= CAx = 0 \\ &\cdots \end{aligned} \quad \Longrightarrow \quad \mathbf{rank}\,\mathcal{O} = \mathbf{rank} \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \neq n$$

- in Kalman canonical form (A_{11}, C_1) is observable subspace

$$\left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} A_{11} & 0 & B_1 \\ A_{21} & A_{22} & B_2 \\ \hline C_1 & 0 & 0 \end{array}\right]$$

  a nonzero vector $(0, x_2)$ with $A_{22}x_2 = \lambda x_2$ is unobservable
- the function-valued operator $\Phi : x_0 \mapsto Ce^{At}x_0$ has $\mathbf{ker}(\Phi) = \mathbf{null}(\mathcal{O})$

Lyapunov theorem with observability

For the state space system $\dot{x} = Ax$, $V(z) = z^T P z$, and

$$\dot{V}(z) = z^T(A^T P + PA)z = -z^T Q z,$$

if $P \succ 0$, $Q \succeq 0$, and (A, Q) is observable, then $x(t) \to 0$.

proof idea. use LaSalle to rule out hidden unstable trajectories

- solution is $x(t) = e^{At}x_0$
- (A, Q) observable $\iff (A, Q^{1/2})$ observable
- if a solution of $\dot{x} = Ax$ is in zero dissipation set $\dot{V}(z) = 0$, then

$$-(e^{At}x_0)^T Q(e^{At}x_0) = -\|Q^{1/2}e^{At}x_0\|_2 = 0 \quad \text{for all } t \geq 0$$

- from PBH this means $x_0 = 0$, hence $x(t) \equiv 0$

Observability zoo

For the Lyapunov equation $A^T P + PA = -Q$

	$P \succ 0$	$P \succeq 0$
$Q \succ 0$	asy. stable	impossible
$Q \succeq 0$	bounded	may have unstable subspaces
$Q \succeq 0$ and (A, Q) obs.	asy. stable	impossible

Dual Lyapunov equation

We have the following LMI equivalence

$$P \succ 0, \quad A^T P + PA \prec 0$$

if and only if

$$Q \succ 0, \quad QA^T + AQ \prec 0$$

for $Q = P^{-1}$.

proof. multiply both sides on the left and right by $Q = P^{-1}$

extremely useful trick in static controller synthesis

Homegeneity

fact. there exists $P \succ 0$, $A^T P + PA \prec 0$ if and only if there exists $\tilde{P}$,

$$\tilde{P} \succeq I, \quad A^T \tilde{P} + \tilde{P} A \prec 0$$

- in practice, we cannot enforce $P \succ 0$ on the computer
- we have for small $\epsilon > 0$,

$$P \succeq \epsilon I, \quad A^T P + PA \prec 0$$

- change variables to $\tilde{P} = P/\epsilon$

$$\tilde{P} \succeq I, \quad A^T \tilde{P} + \tilde{P} A \prec 0$$

- have to be very careful with rank deficiencies in control SDPs