Lecture 6. Foundations of LMIs in System and Control Theory

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CDS270–2: Mathematical Methods in Control and System Engineering

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Logistics

- hw5 due this **Wed, May 6**
  - do an easy problem or CYOA
- hw4 solutions posted online

Riccati differential equation

Recall the Riccati boundary value problem,

$$\begin{aligned} -\dot{P} &= A^T P + PA + Q - PBR^{-1}B^T P \\ P(T) &= Q_T \end{aligned}$$

where $P(t) = P(t)^T \succeq 0$ is the variable.

- $Q, Q_T \succeq 0$ and $R \succ 0$ given
- used in solving finite horizon LQR with terminal matrix Q_T
- steady state solution as $T \to \infty$ is the solution to the ARE,

$$0 = A^T P + PA + Q - PBR^{-1}B^T P$$

- **question.** how do we solve for P in the ARE?

Classical technique

Write down the optimality conditions for finite horizon LQR

$$\begin{array}{ll} \text{minimize} & \frac{1}{2}\int_0^T x(t)^T Q x(t) + u(t)^T R u(t)\, dt + \frac{1}{2} x(T)^T Q_T x(T) \\ \text{subject to} & \dot{x}(t) = Ax(t) + Bu(t), \quad t \in (0, T) \\ & x(0) = z \end{array}$$

- dual "variables" $\nu : [0, T] \to \mathbf{R}^n$
- Lagrangian:

$$L(x, u, \nu) = \frac{1}{2}\int_0^T x(t)^T Q x(t) + u(t)^T R u(t)\, dt + \frac{1}{2} x(T)^T Q_T x(T) + \int_0^T \nu(t)^T (Ax(t) + Bu(t) - \dot{x}(t))\, dt + \nu(0)^T (x(0) - z)$$

Optimality conditions

Lagrangian:

$$L(x,u,\nu) = \frac{1}{2}\int_0^T x(t)^T Qx(t) + u(t)^T Ru(t)\,dt + \frac{1}{2}x(T)^T Q_T x(T) + \int_0^T \nu(t)^T (Ax(t) + Bu(t) - \dot{x}(t))\,dt + \nu(0)^T (x(0) - z)$$

interior. for $t \in (0, T)$ we have:

- integration by parts:

$$\int_0^T \nu(t)^T \dot{x}(t)\,dt = \nu(t)^T x(t)\Big|_{t=0}^{t=T} - \int_0^T \dot{\nu}(t)^T x(t)\,dt$$

- $\nabla_{x(t)} L = Qx(t) + A^T \nu(t) + \dot{\nu}(t) = 0$
- $\nabla_{u(t)} L = Ru(t) + B^T \nu(t) = 0$
- $\nabla_{\nu(t)} L = Ax(t) + Bu(t) - \dot{x}(t) = 0$

Optimality conditions

boundaries.

- $\nabla_{x(T)} L = Q_T x(T) - \nu(T) = 0$
- $\nabla_{\nu(0)} L = x(0) - z = 0$

boundary value problem.

$$\begin{aligned} \dot{x}(t) &= Ax + Bu(t), \quad x(0) = z \\ u(t) &= -R^{-1}B^T\nu(t) \\ -\dot{\nu}(t) &= Qx(t) + A^T\nu(t), \quad \nu(T) = Q_T x(T) \end{aligned}$$

after substituting $u(t) = -R^{-1}B^T\nu(t)$, we obtain

$$\begin{bmatrix} \dot{x} \\ \dot{\nu} \end{bmatrix} = \underbrace{\begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix}}_{H} \begin{bmatrix} x \\ \nu \end{bmatrix}, \quad x(0) = z, \quad \nu(T) = Q_T x(T)$$

Hamiltonian matrix

Consider the Hamiltonian matrix differential equation,

$$\begin{bmatrix} \dot{X}(t) \\ \dot{Y}(t) \end{bmatrix} = \underbrace{\begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix}}_{H} \begin{bmatrix} X(t) \\ Y(t) \end{bmatrix},$$

with matrix variables $X(t), Y(t) \in \mathbf{R}^{n \times n}$

- the matrix $H \in \mathbf{R}^{2n \times 2n}$ is a *Hamiltonian* matrix
- such matrices obey $JH = -H^T J$, where

$$J = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}, \quad (J^{-1} = J^T = -J)$$

- the eigenvalues of H are symmetric about real and imaginary axes
  proof. H is real, and $H^T v = \lambda v$ implies $HJv = -\lambda Jv$

Relationship between Riccati and Hamiltonian ODEs

If $X(t), Y(t) \in \mathbf{R}^{n\times n}$ obey the Hamiltonian ODE

$$\begin{bmatrix} \dot{X}(t) \\ \dot{Y}(t) \end{bmatrix} = \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix} \begin{bmatrix} X(t) \\ Y(t) \end{bmatrix},$$

then $P(t) = Y(t)X(t)^{-1}$ obeys the Riccati ODE,

$$-\dot{P} = A^T P + PA + Q - PBR^{-1}B^T P.$$

proof.

$$\begin{aligned} -\dot{P} &= -(\dot{Y}X^{-1} + Y(\dot{X^{-1}})) \\ &= -\dot{Y}X^{-1} + YX^{-1}\dot{X}X^{-1} \quad (\text{using } \dot{X}X^{-1} + X(\dot{X^{-1}}) = 0) \\ &= (QX + A^T Y)X^{-1} + YX^{-1}(AX - BR^{-1}B^T Y)X^{-1} \\ &= A^T P + PA + Q - PBR^{-1}B^T P. \end{aligned}$$

Spectrum of the Hamiltonian matrix

The matrix P defines a similarity transformation,

$$\begin{bmatrix} I & 0 \\ -P & I \end{bmatrix} \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix} \begin{bmatrix} I & 0 \\ P & I \end{bmatrix}$$
$$= \begin{bmatrix} A - BR^{-1}B^TP & -BR^{-1}B^T \\ -(A^TP + PA + Q - PBR^{-1}B^TP) & -(A - BR^{-1}B^TP)^T \end{bmatrix}$$

and if P further satisfies the ARE, then with $K = -R^{-1}B^TP$, this equals

$$\begin{bmatrix} A + BK & -BR^{-1}B^T \\ 0 & -(A + BK)^T \end{bmatrix}.$$

Thus if (A, B) is controllable, the eigenvalues of H are related to the closed loop eigenvalues by

$$\mathbf{spec}(H) = \mathbf{spec}(A + BK) \cup \mathbf{spec}(-(A + BK)).$$

Solving the ARE

If $A + BK$ (stable) is diagonalizable,

$$T^{-1}(A + BK)T = \Lambda,$$

then

$$\begin{aligned}
&\begin{bmatrix} T^{-1} & 0 \\ 0 & T^T \end{bmatrix} \begin{bmatrix} I & 0 \\ -P & I \end{bmatrix} H \begin{bmatrix} I & 0 \\ P & I \end{bmatrix} \begin{bmatrix} T & 0 \\ 0 & T^{-T} \end{bmatrix} \\
&\qquad = \begin{bmatrix} T^{-1} & 0 \\ 0 & T^T \end{bmatrix} \begin{bmatrix} A + BK & -BR^{-1}B^T \\ 0 & -(A + BK)^T \end{bmatrix} \begin{bmatrix} T & 0 \\ 0 & T^{-T} \end{bmatrix} \\
&\qquad = \begin{bmatrix} \Lambda & -T^{-1}BR^{-1}B^T T^{-T} \\ 0 & -\Lambda \end{bmatrix},
\end{aligned}$$

hence

$$H \begin{bmatrix} T \\ PT \end{bmatrix} = \begin{bmatrix} T \\ PT \end{bmatrix} \Lambda.$$

Classical algorithm

We wish to solve[1] the ARE

$$A^T P + PA + Q - PBR^{-1}B^T P = 0, \quad P \succeq 0$$

1. form the Hamiltonian matrix

$$H = \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix}$$

2. find eigenvectors $v_1, \ldots, v_n$ of H corresponding to the n stable eigenvalues, and partition as

$$\begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix} = \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} T \\ PT \end{bmatrix} \in \mathbf{R}^{2n \times n}$$

3. the unique positive semidefinite solution to the ARE is given by

$$P := YX^{-1}$$

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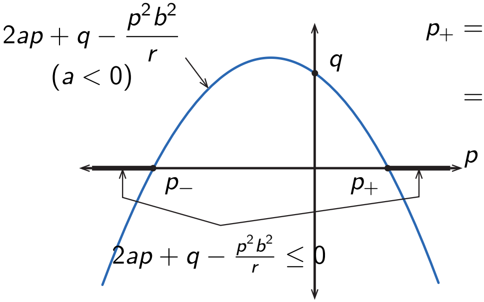
[1] for more general discussion, involving Jordan forms, see JE Potter, "Matrix Quadratic Solutions," *J. SIAM Appl. Math.* 14(3):496–501, 1966

Single dimension

In the $n = 1$ case, the ARE is

$$2ap + q - \frac{p^2 b^2}{r} = 0$$

where the variable is $p \in \mathbf{R}$. We seek the positive solution, p_+.



$$p_+ = \inf\{p \mid p \geq 0, 2ap + q - \frac{p^2 b^2}{r} \leq 0\}$$

$$= \sup\{p \mid p \geq 0, 2ap + q - \frac{p^2 b^2}{r} \geq 0\}$$

Riccati inequality (nonstandard)

In the full matrix case, change the equality to (matrix) inequality,

$$A^T P + PA + Q - PBR^{-1}B^T P \preceq 0 \tag{1}$$

fact. if (A, B) controllable and $(A, Q^{1/2})$ observable, and $P_{\text{are}} \succeq 0$ with $A^T P_{\text{are}} + P_{\text{are}} A + Q - P_{\text{are}} BR^{-1}B^T P = 0$, then P_{are} is minimal, in the sense that for any P satisfying (1) we have

$$P_{\text{are}} \preceq P.$$

Riccati equation: nonconvex approach

This suggests that the ARE is a "closed-form" solution to the nonconvex problem

$$\begin{array}{ll} \text{minimize} & x(0)^T P x(0) \\ \text{subject to} & A^T P + PA + Q - PBR^{-1}B^T P \preceq 0 \\ & P \succeq 0 \end{array}$$

danger. we *cannot* use a Schur complement here,

$$A^T P + PA + Q - PBR^{-1}B^T P \preceq 0 \quad \not\Leftrightarrow \quad \begin{bmatrix} A^T P + PA + Q & PB \\ B^T P & R \end{bmatrix} \preceq 0$$

(because $R \not\prec 0$)

Riccati inequality (standard)

The "other" Riccati inequality is much more common,

$$A^T P + PA + Q - PBR^{-1}B^T P \succeq 0 \tag{2}$$

fact. if (A, B) controllable and $(A, Q^{1/2})$ observable, and $P_{\text{are}} \succeq 0$ with $A^T P_{\text{are}} + P_{\text{are}} A + Q - P_{\text{are}} BR^{-1}B^T P = 0$, then P_{are} is maximal, in the sense that for any P satisfying (2) we have

$$P \preceq P_{\text{are}}.$$

Riccati equation: LMI approach

The standard Riccati inequality leads to the convex problem:

$$\begin{array}{ll} \text{maximize} & x(0)^T P x(0) \\ \text{subject to} & A^T P + PA + Q - PBR^{-1}B^T P \succeq 0 \\ & P \succeq 0. \end{array}$$

Since $R \succ 0$, use a Schur complement to obtain the equivalent SDP:

$$\begin{array}{ll} \text{maximize} & x(0)^T P x(0) \\ \text{subject to} & \begin{bmatrix} A^T P + PA + Q & PB \\ B^T P & R \end{bmatrix} \succeq 0 \\ & P \succeq 0 \end{array}$$

Aside: SDP duality

Consider the SDP in inequality form

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & x_1 F_1 + \cdots + x_n F_n + G \preceq 0 \end{array}$$

where $x \in \mathbf{R}^n$ is the variable.

- dual variable $Z = Z^T$
- Lagrangian:

$$\begin{aligned} L(x, Z) &= c^T x + \mathbf{Tr}((x_1 F_1 + \cdots + x_n F_n + G)Z) \\ &= x_1(c_1 + \mathbf{Tr}(F_1 Z)) + \cdots + x_n(c_n + \mathbf{Tr}(F_n Z)) + \mathbf{Tr}(GZ), \end{aligned}$$

  which is affine in $x \in \mathbf{R}^n$

- dual function:

$$g(Z) = \inf_x L(x, Z) = \begin{cases} \mathbf{Tr}(GZ), & \mathbf{Tr}(F_i Z) + c_i = 0, \text{ for all } i = 1, \ldots, n \\ -\infty, & \text{otherwise} \end{cases}$$

Aside: SDP duality

primal SDP:

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & x_1 F_1 + \cdots + x_n F_n + G \preceq 0 \end{array}$$

dual SDP:

$$\begin{array}{ll} \text{maximize} & \mathbf{Tr}(GZ) \\ \text{subject to} & \mathbf{Tr}(F_i Z) + c_i = 0, \quad i = 1, \ldots, n \\ & Z \succeq 0 \end{array}$$

Strong duality obtains if primal is strictly feasible, *i.e.*, there is an $x \in \mathbf{R}^n$,

$$x_1 F_1 + \cdots + x_n F_n + G \prec 0.$$

Taking the dual

We wish to find the dual of the SDP

$$
\begin{array}{ll}
\text{maximize} & x(0)^T P x(0) \\
\text{subject to} & \begin{bmatrix} A^T P + PA + Q & PB \\ B^T P & R \end{bmatrix} \succeq 0 \\
& P \succeq 0
\end{array}
$$

- dual variables associated with the two constraints:

$$
\begin{bmatrix} \bar{Q} & Y^T \\ Y & Z \end{bmatrix} = \begin{bmatrix} \bar{Q}^T & Y^T \\ Y & Z^T \end{bmatrix} \succeq 0, \quad W = W^T \succeq 0
$$

- Lagrangian (note the signs):

$$
\begin{aligned}
L(P, \bar{Q}, Y, Z, W) &= x(0)Px(0) \\
&\quad + \mathbf{Tr} \begin{bmatrix} A^T P + PA + Q & PB \\ B^T P & R \end{bmatrix} \begin{bmatrix} \bar{Q} & Y^T \\ Y & Z \end{bmatrix} + \mathbf{Tr}(PW)
\end{aligned}
$$

Taking the dual

Simpifying the Lagrangian,

$$
\begin{aligned}
&L(P, \bar{Q}, Y, Z, W) = \\
&= x(0)Px(0) + \mathbf{Tr}\begin{bmatrix} A^T P + PA + Q & PB \\ B^T P & R \end{bmatrix}\begin{bmatrix} \bar{Q} & Y^T \\ Y & Z \end{bmatrix} + \mathbf{Tr}(PW) \\
&= \mathbf{Tr}\left\{XP + (A^T P + PA + Q)\bar{Q} + PBY + B^T P Y^T + RZ + PW\right\} \\
&= \mathbf{Tr}\left\{(X + \bar{Q}A^T + A\bar{Q} + BY + Y^T B^T + W)P\right\} + \mathbf{Tr}(Q\bar{Q} + RZ),
\end{aligned}
$$

where $X = x(0)x(0)^T$ and we used the cyclic property of $\mathbf{Tr}(\cdot)$

The dual function is a supremum (when primal is “maximize”),

$$
g(\bar{Q}, Y, Z, W) = \sup_P L(P, \bar{Q}, Y, Z, W)
$$

$$
= \begin{cases} \mathbf{Tr}(Q\bar{Q} + RZ), & X + \bar{Q}A^T + A\bar{Q} + BY + Y^T B^T + W = 0 \\ \infty, & \text{otherwise} \end{cases}
$$

Taking the dual

Thus the dual is an SDP

$$\begin{array}{ll} \text{minimize} & \mathbf{Tr}(Q\bar{Q} + RZ) \\ \text{subject to} & X + \bar{Q}A^T + A\bar{Q} + BY + Y^T B^T + W = 0 \\ & \begin{bmatrix} \bar{Q} & Y^T \\ Y & Z \end{bmatrix} \succeq 0 \end{array}$$

- objective is the LQR cost
- primal constraint $P \succeq 0$ is automatically satisfied, so $W = 0$
- the dual variable turns out to be the state-input *Gram matrix*

$$\begin{bmatrix} \bar{Q} & Y^T \\ Y & Z \end{bmatrix} = \int_0^\infty \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \begin{bmatrix} x^T(t) & u^T(t) \end{bmatrix} dt$$

(for technical considerations, see Seungil's thesis)

State-input Gram matrix

Consider the quantity

$$\begin{aligned}\frac{d}{dt}(x(t)x(t)^T) &= \dot{x}(t)x(t)^T + x(t)\dot{x}(t)^T \\ &= (Ax + Bu)x^T + x(Ax + Bu)^T \\ &= xx^T A^T + Axx^T + Bux^T + xu^T B^T.\end{aligned}$$

Take the integral of both sides

$$\underbrace{x(\infty)x(\infty)^T}_{=0} - \underbrace{x(0)x(0)^T}_{=X} = \int_0^\infty xx^T A^T + Axx^T + Bux^T + xu^T B^T \, dt,$$

which is the equality constraint

$$-X = \bar{Q}A^T + A\bar{Q} + BY + Y^T B^T = 0,$$

where $\bar{Q} \triangleq \int_0^\infty xx^T \, dt$, $Y \triangleq \int_0^\infty ux^T \, dt$, and $Z \triangleq \int_0^\infty uu^T \, dt$.