Lecture 7. LMI approaches to H_2, H_∞ problems

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CDS270–2: Mathematical Methods in Control and System Engineering

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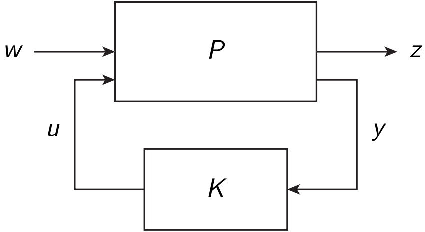
Logistics

- hw6 due this **Wed, May 13**
  - do an easy problem or CYOA
  - use catalog with date stamp $\geq$ 05/06/2015
  - part 3(d): uses `Matlab` and `CVX`
- hw5 solutions posted online
- reading: lmibook Ch 4–6

Control system



for a plant P and controller K we define the following signals

- *exogenous inputs*: $w \in \mathbf{R}^{n_w}$
- *actuator inputs*: $u \in \mathbf{R}^{n_u}$
- *regulated outputs*: $z \in \mathbf{R}^{n_z}$
- *sensed outputs*: $y \in \mathbf{R}^{n_y}$

Common signal measures

Let $y : [0, \infty) \to \mathbf{R}^n$ be a signal

$\mathbf{L}_\infty$ (peak) norm:

$$\|y\|_\infty = \max_{1 \le i \le n} \|y_i\|_\infty = \sup_{t \ge 0} \max_{1 \le i \le n} |y_i(t)|$$

$\mathbf{L}_2$ (total energy) norm:

$$\begin{aligned} \|y\|_2 &= \left(\int_0^\infty y(t)^T y(t)\, dt \right)^{1/2} \\ &= \left(\frac{1}{2\pi} \int_{-\infty}^\infty \hat{y}(j\omega)^* \hat{y}(j\omega)\, d\omega \right)^{1/2} \quad \text{(Parseval)} \end{aligned}$$

root-mean-square seminorm:

$$\|y\|_{\mathrm{rms}} = \left(\lim_{T \to \infty} \frac{1}{T} \int_0^T y(t)^T y(t)\, dt \right)^{1/2}$$

Common system norms

Let H be a system with impulse response matrix $h(t)$

$\mathbf{H}_2$ (RMS response to white noise):

$$\|H\|_2 = \left(\mathbf{Tr} \, \frac{1}{2\pi} \int_0^\infty H(j\omega)^* H(j\omega) \, d\omega \right)^{1/2}$$

$$= \left(\frac{1}{2\pi} \sum_{i=1}^{n} \int_{-\infty}^{\infty} \sigma_i (H(j\omega))^2 \, d\omega \right)^{1/2}$$

$$= \left(\mathbf{Tr} \int_0^\infty h(t)^T h(t) \, dt \right)^{1/2}$$

$\mathbf{H}_\infty$ (RMS or $\mathbf{L}_2$ gain):

$$\|H\|_\infty = \sup_{\|w\|_2 \neq 0} \frac{\|Hw\|_2}{\|w\|_2} = \sup_\omega \sigma_{\max}(H(j\omega))$$

Computing H_2-norm

Consider the system

$$H: \quad \dot{x} = Ax + B_w w, \quad z = C_z x, \quad x(0) = 0.$$

- impulse response is $h(t) = C_z e^{At} B_w$, follows from $w(t) = \delta(t)$ in

$$y(t) = C_z \int_{0^-}^{t} e^{A(t-\tau)} B_w w(\tau)\, d\tau.$$

- substitute impulse response into

$$\begin{aligned} \|H\|_2^2 &= \mathbf{Tr}\left(\int_0^\infty h(t)^T h(t)\, dt \right) \\ &= \mathbf{Tr}\left(B_w^T \int_0^\infty e^{A^T t} C_z^T C_z e^{At}\, dt\, B_w \right) \\ &= \mathbf{Tr}(B_w^T W_{\mathrm{obs}} B_w) \end{aligned}$$

Computing H_2-norm

The $\mathbf{H}_2$ norm of the system satisfies

$$\|H\|_2^2 = \mathbf{Tr}(B_w^T W_{\text{obs}} B_w),$$

where W_{obs} is the observability Gramian, given by

$$W_{\text{obs}} \stackrel{\Delta}{=} \int_0^\infty e^{A^T t} C_z^T C_z e^{At}\, dt,$$

or equivalently, the solution to the Lyapunov equation

$$A^T W_{\text{obs}} + W_{\text{obs}} A + C_z^T C_z = 0.$$

Controllability perspective

Using the cyclic property of $\mathbf{Tr}(\cdot)$,

$$\begin{aligned}
\|H\|_2^2 &= \mathbf{Tr}\left(\int_0^\infty h(t)^T h(t)\, dt\right) \\
&= \mathbf{Tr}\left(\int_0^\infty B_w^T e^{A^T t} C_z^T C_z e^{At} B_w\, dt\right) \\
&= \mathbf{Tr}\left(C_z \int_0^\infty e^{At} B_w B_w^T e^{A^T t}\, dt\, C_z^T\right) \\
&= \mathbf{Tr}(C_z W_{\text{contr}} C_z^T),
\end{aligned}$$

where W_{contr} is the controllability Gramian,

$$W_{\text{contr}} \stackrel{\Delta}{=} \int_0^\infty e^{A^T t} B_w B_w^T e^{At}\, dt,$$

or equivalently, the solution to the Lyapunov equation

$$W_{\text{contr}} A^T + A W_{\text{contr}} + B_w B_w^T = 0.$$

Lagrange duality

In fact, the following two SDPs are Lagrange duals of each other,

$$\begin{array}{ll} \text{minimize} & \mathbf{Tr}(C_z Q C_z^T) \\ \text{subject to} & Q \succeq 0, \\ & QA^T + AQ + B_w B_w^T \preceq 0 \end{array} \quad (1)$$

and

$$\begin{array}{ll} \text{maximize} & \mathbf{Tr}(B_w^T P B_w) \\ \text{subject to} & P \succeq 0 \\ & A^T P + PA + C_z^T C_z \succeq 0 \end{array} \quad (2)$$

- if strong duality obtains, then

$$\mathbf{Tr}(C_z Q^\star C_z^T) = \mathbf{Tr}(B_w^T P^\star B_w)$$

- strong duality is implied by strict feasibility of (1) or (2) ... which happens if A is stable.

Strong duality in H_2 SDP

link to H_2 norm. If strong duality obtains in (1) and (2), and either $P^\star \succ 0$ or $Q^\star \succ 0$, then

$$\|H\|_2^2 = \mathbf{Tr}(C_z Q^\star C_z^T) = \mathbf{Tr}(B_w^T P^\star B_w).$$

proof. by strong duality, we have

$$\begin{aligned}
\mathbf{Tr}(C_z Q^\star C_z^T) &= \mathbf{Tr}(B_w^T P^\star B_w) \\
&= \inf_{Q \succeq 0} \mathbf{Tr}(C_z Q C_z^T) + \mathbf{Tr}((QA^T + AQ + B_w B_w^T) P^\star) \\
&\leq \mathbf{Tr}(C_z Q^\star C_z^T) + \mathbf{Tr}((Q^\star A^T + AQ^\star + B_w B_w^T) P^\star) \\
&\leq \mathbf{Tr}(C_z Q^\star C_z^T),
\end{aligned}$$

thus all the inequalities hold with equality. If $P^\star \succ 0$, then

$$Q^\star A^T + AQ^\star + B_w B_w^T = 0,$$

i.e., $Q^\star = W_{\text{contr}}$. (If $Q^\star \succ 0$ instead, we get $P^\star = W_{\text{obs}}$.)

Strong duality in H_2 SDP

fact. If A is (Hurwitz) stable, then strong duality obtains in (1) and (2).

proof. if A is Hurwitz stable, there exists a matrix $Q_0 \succ 0$ such that

$$Q_0 A^T + AQ_0 + (\epsilon I + B_w B_w^T) = 0,$$

where $\epsilon > 0$ is any positive number. Therefore

$$Q_0 A^T + AQ_0 + B_w B_w^T = -\epsilon I \prec 0,$$

meaning (1) is strictly feasible. By Slater's condition, we have strong duality.

Strong duality in H_2 SDP

fact. Suppose A is (Hurwitz) stable.

- if (A, B_w) is controllable, then $Q^\star \succ 0$ and $P^\star = W_{\text{obs}}$
- if (A, C_z) is observable, then $P^\star \succ 0$ and $Q^\star = W_{\text{contr}}$

proof (first statement). since $Q^\star$ is feasible in (1), it is a generalized controllability Gramian, so it satisfies

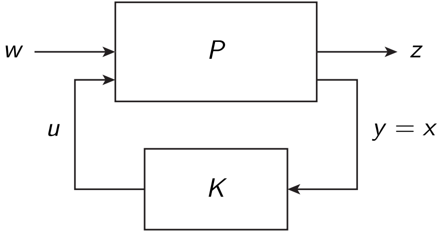
$$Q^\star \succeq W_{\text{contr}},$$

and because (A, B_w) is controllable with A Hurwitz, we further have $W_{\text{contr}} \succ 0$. Therefore $Q^\star \succeq W_{\text{contr}} \succ 0$. From strong duality,

$$\mathbf{Tr}((A^T P^\star + P^\star A + C_z^T C_z) Q^\star) = 0$$

implies $A^T P^\star + P^\star A + C_z^T C_z = 0$, thus, $P^\star = W_{\text{obs}}$, as required.

$\mathbf{H}_2$ state feedback synthesis problem



Given the system

$$\dot{x} = Ax + B_u u + B_w w, \quad z = C_z x + D_{zu} u, \quad x(0) = 0$$

find a state feedback input $u = Kx$ to minimize the w-to-z $\mathbf{H}_2$ norm.

Interpretation: calculating the w-to-z $\mathbf{H}_2$ norm

For constant state-feedback $u = Kx$, the closed loop system is

$$\dot{x} = (A + B_u K)x + B_w w$$
$$z = (C_z + D_{zu} K)x$$

thus the w-to-z $\mathbf{H}_2$ norm is simply the energy of the output

$$E = \int_0^\infty z(t)^T z(t)\, dt,$$

with the choice $w(t) = \delta(t)$

Calculating the w-to-z $\mathbf{H}_2$ norm

Choosing $w(t) = \delta(t)$ for the system

$$\begin{aligned} \dot{x} &= (A + B_u K)x + B_w w \\ z &= (C_z + D_{zu} K)x \\ x(0) &= 0 \end{aligned}$$

is the same as having a nonzero initial condition $x(0) = B_w$

$$\begin{aligned} \dot{x} &= (A + B_u K)x \\ z &= (C_z + D_{zu} K)x \\ x(0) &= B_w, \end{aligned}$$

proof.

$$\begin{aligned} x(t) &= e^{(A+B_u K)t} \cdot 0 + \int_{0^-}^{t} e^{(A+B_u K)(t-\tau)} B_w \cdot \delta(\tau)\, d\tau \\ &= e^{(A+B_u K)t} \cdot B_w \end{aligned}$$

$\mathbf{H}_2$ state feedback synthesis

In the language of the LMI (1), the w-to-z $\mathbf{H}_2$ norm is given by solving the problem

$$\begin{array}{ll} \text{minimize} & \mathbf{Tr}\left((C_z + D_{zu}K)Q(C_z + D_{zu}K)^T\right) \\ \text{subject to} & Q \succeq 0 \\ & Q(A + B_uK)^T + (A + B_uK)Q + B_wB_w^T \preceq 0 \end{array}$$

- the objective is simulatenously the $\mathbf{H}_2$ norm, and the output energy E we wish to minimize.
- if $A + B_uK$ is stable, strong duality obtains
- if K is a variable, the problem is nonconvex

Lyapunov function perspective

output energy minimization

$$\dot{x} = Ax + B_u u, \quad z = C_z x + D_{zu} u \tag{3}$$

where (A, B, C) is minimal, $D_{zu}^T D_{zu} \succ 0$, and $D_{zu}^T C_z = 0$. Given an initial condition $x(0)$ find an input $u = Kx$ to minimize the output energy

$$E = \int_0^\infty z(t)^T z(t)\, dt.$$

fact. if there exists a storage function $V(x) = x^T Px$, $P \succ 0$, and

$$\frac{d}{dt} V(x) \leq -z^T z, \quad \text{for all } z, x, u = Kx \text{ satisfying (3)},$$

then $x(0)^T Px(0)$ is an upper bound on E.

Lyapunov argument

Integrate $\frac{d}{dt}V(x) \leq -z^T z$ to get

$$V(x(T)) - V(x(0)) \leq -\int_0^T z^T z \, dt, \qquad \text{for all } T \geq 0.$$

Since $V(x(T)) \geq 0$, and this is true for all T, we therefore have

$$V(x(0)) \geq \int_0^\infty z^T z \, dt \quad (= E).$$

- $V(x(0)) = x(0)^T P x(0)$ is an upper bound on the output energy
- to make output energy small, we minimize this upper bound

Solution to problem

We wish to minimize the upper bound $x(0)^T P x(0)$ subject to the dissipation condition:

$$\frac{d}{dt} V(x) \leq -z^T z, \quad \text{for all } z, x, u = Kx \text{ satisfying (3)}$$

$$\begin{aligned}
&\iff \dot{x}^T P x + x^T P \dot{x} \leq -z^T z, \text{ for all } z, x, u = Kx \text{ satisfying (3)} \\
&\iff x^T (A + B_u K)^T P x + x^T P (A + B_u K) x \\
&\qquad\qquad \leq -x^T (C_z + D_{zu} K)^T (C_z + D_{zu} K) x, \text{ for all } x \in \mathbf{R}^n \\
&\iff A^T P + PA + K^T B_u^T P + P B_u K + C_z^T C_z + K^T (D_{zu}^T D_{zu}) K \preceq 0 \\
&\iff QA^T + AQ + QK^T B_u^T + B_u K Q \\
&\qquad\qquad + (C_z Q)^T (C_z Q) + QK^T (D_{zu}^T D_{zu}) KQ \preceq 0
\end{aligned}$$

where in the last step we multiplied on the left and right by $Q = P^{-1}$

State feedback trick

If we define the variable $Y = KQ$, then we have

$$QA^T + AQ + QK^T B_u^T + B_u KQ + (C_z Q)^T (C_z Q) + QK^T (D_{zu}^T D_{zu}) KQ \preceq 0$$

$$\iff$$

$$QA^T + AQ + Y^T B_u^T + B_u Y^T + (C_z Q)^T (C_z Q) + Y^T (D_{zu}^T D_{zu}) Y \preceq 0$$

Taking a Schur complement gives the LMI

$$\begin{bmatrix} QA^T + AQ + Y^T B_u^T + B_u Y^T & (C_z Q + D_{zu} Y)^T \\ (C_z Q + D_{zu} Y) & -I \end{bmatrix} \preceq 0$$

Output energy minimization summary

state feedback synthesis. solve the problem

$$
\begin{array}{ll}
\text{minimize} & x(0)^T Q^{-1} x(0) \\
\text{subject to} & \begin{bmatrix} QA^T + AQ + Y^T B_u^T + B_u Y^T & (C_z Q + D_{zu} Y)^T \\ (C_z Q + D_{zu} Y) & -I \end{bmatrix} \preceq 0, \\
& Q \succ 0
\end{array}
$$

with variables $Q = Q^T \in \mathbf{R}^{n \times n}$ and $Y \in \mathbf{R}^{n_u \times n}$

solution.

- the optimal value $x(0)^T (Q^\star)^{-1} x(0)$ is an upper bound on the energy

$$E = \int_0^\infty z^T z \, dt$$

- the optimal state feedback is $K = Y^\star (Q^\star)^{-1}$

Bound on $\mathbf{H}_\infty$-norm

Consider the system

$$H: \quad \dot{x} = Ax + B_w w, \quad z = C_z x, \quad x(0) = 0. \tag{4}$$

If there exists a storage function $V : \mathbf{R}^n \to \mathbf{R}_+$ such that

$$\dot{V} + z^T z - \gamma w^T w \leq 0, \quad V(0) = 0$$

for all x and w satisfying (4), then $\|H\|_\infty^2 \leq \gamma$.

proof. integrate to obtain

$$\underbrace{\int_0^\infty \dot{V}(x(t))\, dt}_{\geq 0} + \|z\|_2^2 \leq \gamma \|w\|_2^2.$$

Quadratic storage function

For $V(x) = x^T P x$, $P \succ 0$, the condition

$$\dot{V} + z^T z - \gamma w^T w \leq 0$$

for all x and w satisfying (4), is the same as

$$(Ax + B_w w)^T P x + x^T P(Ax + B_w w) + x^T (C_z^T C_z) x - \gamma w^T w \leq 0.$$

This translates to the LMI:

$$P \succ 0, \quad \begin{bmatrix} A^T P + PA + C_z^T C_z & PB_w \\ B_w^T P & -\gamma I \end{bmatrix} \preceq 0.$$

Calculating the $\mathbf{H}_\infty$-norm of a system

Now consider minimizing the upper bound γ,

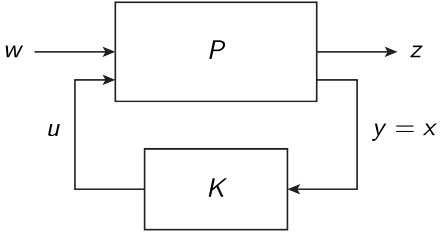
$$\begin{array}{ll} \text{minimize} & \gamma \\ \text{subject to} & P \succ 0 \\ & \begin{bmatrix} A^T P + PA + C_z^T C_z & PB_w \\ B_w^T P & -\gamma I \end{bmatrix} \preceq 0. \end{array}$$

fact. (Kalman–Yakubovich–Popov) the optimal solution to the problem above is $\gamma^\star = \|H\|_\infty^2 = \|C_z(sI - A)^{-1} B_w\|_\infty^2$.

- quadratic storage function is enough
- worst case gain is the $\mathbf{H}_\infty$-norm (suitably squared):

$$\|z\|_2^2 \le \gamma \|w\|_2^2 \quad \iff \quad \|H\|_\infty^2 = \sup_{\|w\|_2 \neq 0} \frac{\|z\|_2^2}{\|w\|_2^2} \le \gamma$$

$\mathbf{H}_\infty$ state feedback synthesis problem



Given the system

$$\dot{x} = Ax + B_u u + B_w w, \quad z = C_z x + D_{zu} u, \quad x(0) = 0$$

find a state feedback input $u = Kx$ to minimize the w-to-z $\mathbf{H}_\infty$ norm.

$\mathbf{H}_\infty$ state feedback solution

Once again, we minimize the $\mathbf{H}_\infty$ norm upper bound for the closed loop system

$$\begin{aligned} \dot{x} &= (A + B_u K)x + B_w w \\ z &= (C_z + D_{zu} K)x \\ x(0) &= 0, \end{aligned}$$

i.e., we wish to solve the nonconvex problem

$$\begin{array}{ll} \text{minimize} & \gamma \\ \text{subject to} & P \succ 0 \end{array}$$
$$\begin{bmatrix} (A + B_u K)^T P + P(A + B_u K) + (C_z + D_{zu} K)^T (C_z + D_{zu} K) & P B_w \\ B_w^T P & -\gamma I \end{bmatrix} \preceq 0.$$

$\mathbf{H}_\infty$ state feedback synthesis solution

A sequence of manipulations with $Y = KQ$ gives the equivalent problem

$$
\begin{array}{ll}
\text{minimize} & \gamma \\
\text{subject to} & Q \succ 0
\end{array}
$$

$$
\begin{bmatrix} AQ + QA^T + B_u Y + Y^T B_u^T + B_w B_w^T & (C_z Q + D_{zu} Y)^T \\ C_z Q + D_{zu} Y & -\gamma I \end{bmatrix} \preceq 0.
$$

- optimal controller is $K = Y^\star (Q^\star)^{-1}$

Time domain properties

Let's explore the input to state properties of the LDI

$$\dot{x} = A(t)x + B_w(t)w, \quad \begin{bmatrix} A(t) & B_w(t) \end{bmatrix} \in \Omega. \tag{5}$$

Consider the following subsets of $\mathbf{R}^n$:

- reachable set with unit-energy input:

$$\mathcal{R}_{\text{ue}} = \left\{ x(T) \in \mathbf{R}^n \;\middle|\; \begin{array}{c} x, w \text{ satisfy (5)}, \quad x(0) = 0, \\ \int_0^T w^T w \, dt \leq 1, \quad T \geq 0 \end{array} \right\}$$

- reachable set with unit-peak input:

$$\mathcal{R}_{\text{up}} = \left\{ x(T) \in \mathbf{R}^n \;\middle|\; \begin{array}{c} x, w \text{ satisfy (5)}, \quad x(0) = 0, \\ w^T w \leq 1, \quad T \geq 0 \end{array} \right\}$$

- ellipsoid parameterized by P:

$$\mathcal{E} = \{ x \in \mathbf{R}^n \mid x^T P x \leq 1 \}$$

Bounding $\mathcal{R}_{\text{ue}}$ with an ellipsoid

The following inclusion holds:

- $\mathcal{R}_{\text{ue}} \subseteq \mathcal{E}$ if there exists $V(x) = x^T P x$, $P \succ 0$, such that

$$\dot{V}(x) \leq w^T w, \quad \text{for all } x, w \text{ satisfying (5).}$$

proof. suppose such V exists and $x(T) \in \mathcal{R}_{\text{ue}}$, where (recall)

$$\mathcal{R}_{\text{ue}} = \left\{ x(T) \in \mathbf{R}^n \;\middle|\; \begin{array}{c} x, w \text{ satisfy (5)}, \quad x(0) = 0, \\ \int_0^T w^T w \, dt \leq 1, \quad T \geq 0 \end{array} \right\}.$$

For this landing state $x(T)$ and any input w that gets us there,

$$V(x(T)) - V(x(0)) = \int_0^T \dot{V}(x(t)) \, dt \leq \int_0^T w^T w \, dt \leq 1,$$

therefore $x(T)^T P x(T) \leq 1$, *i.e.*, $x(T) \in \mathcal{E}$

Bounding $\mathcal{R}_{\text{up}}$ with an ellipsoid

The following inclusion holds:

- $\mathcal{R}_{\text{up}} \subseteq \mathcal{E}$ if there exists $V(x) = x^T P x$, $P \succ 0$, such that

$$\dot{V}(x) \leq 0, \quad \text{for all } x, w \text{ satisfying (5)}, w^T w \leq 1, \text{ and } V(x) \geq 1.$$

proof idea. for any admissible input (satisfying pointwise unit peak constraints $w^T w \leq 1$), as soon as $V(x(T)) \geq 1$ at some time T, then for all times t thereafter,

$$V(x(t)) \leq V(x(T)), \quad \text{for all } t \geq T.$$

In other words, trajectories with an admissible input cannot exit the 1-sublevel set $\{x \mid V(x) \leq 1\}$.

Ellipsoidal bounds on reachable sets

For LTI systems $\Omega = [A, B]$, the reachability conditions can be rewritten

- $\mathcal{R}_{\text{ue}} \subseteq \mathcal{E}$: there exists $V(x) = x^T P x$, $P \succ 0$, such that

$$\dot{V}(x) \leq w^T w, \quad \text{for all } x, w \text{ satisfying (5).}$$

  is equivalent to feasibility of

$$P \succ 0, \quad \begin{bmatrix} A^T P + PA & PB_w \\ B_w^T P & -I \end{bmatrix} \preceq 0.$$

- $\mathcal{R}_{\text{up}} \subseteq \mathcal{E}$: there exists $V(x) = x^T P x$, $P \succ 0$, such that

$$\dot{V}(x) \leq 0, \quad \text{for all } x, w \text{ satisfying (5)}, w^T w \leq 1, \text{ and } V(x) \geq 1.$$

  is implied by feasibility of the bilinear matrix inequality

$$P \succ 0, \quad \alpha \geq 0, \quad \begin{bmatrix} A^T P + PA + \alpha P & PB_w \\ B_w^T P & -\alpha I \end{bmatrix} \preceq 0.$$