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Bounded-real and positive-real lemmas

The following two lemmas relate the frequency domain characteristics of a signal to the feasibility of a certain LMI, and the solvability of a certain ARE. The positive-real version originally appeared in [Yak62], and both are instances of what is now called the Kalman–Yakubovich–Popov (KYP) lemma.

Let $A \in \mathbf{R}^{n\times n}$, $B \in \mathbf{R}^{n\times p}$, $C \in \mathbf{R}^{p\times n}$ and $D \in \mathbf{R}^{p\times p}$ be given matrices corresponding to a system with the same number p of inputs as outputs, and n internal states.

1 Bounded-real lemma

Assume that A is stable, (A, B, C) is minimal, and $D^T D \prec I$. The following are equivalent:

1. The system
$$\dot{x} = Ax + Bu, \quad y = Cx + Du, \quad x(0) = 0$$
is nonexpansive, *i.e.*, satisfies
$$\int_0^T y(t)^T y(t)\, dt \leq \int_0^T u(t)^T u(t)\, dt$$
for all u and $T \geq 0$.

2. The transfer matrix $H(s) = C(sI - A)^{-1}B + D$ is bounded-real, *i.e.*,
$$H(s)^* H(s) \preceq I$$
for all s with $\mathrm{Re}(s) > 0$, or equivalently, the $\mathbf{H}_\infty$ norm is bounded, $\|H(s)\|_\infty \leq 1$, where
$$\|H(s)\|_\infty = \sup\{\|H(s)\|_2 \mid \mathrm{Re}(s) > 0\}$$

3. The LMI
$$P \succ 0, \quad \begin{bmatrix} A^T P + PA + C^T C & PB + C^T D \\ B^T P + D^T C & D^T D - I \end{bmatrix} \preceq 0$$
in the variable $P = P^T$ is feasible. This corresponds to existence of a quadratic storage function $V(x) = x^T P x$ that satisfies
$$\dot{V} + y^T y - u^T u \leq 0.$$

4. There exists a real matrix $P = P^T$ satsifying the ARE

$$A^T P + PA + C^T C + (PB + C^T D)(I - D^T D)^{-1}(PB + C^T D)^T = 0.$$

5. The Hamiltonian matrix

$$M = \begin{bmatrix} A + B(I - D^T D)^{-1} D^T C & B(I - D^T D)^{-1} B^T \\ -C^T (I - D^T D)^{-1} C & -A^T - C^T D(I - D^T D)^{-1} B^T \end{bmatrix}$$

has no imaginary eigenvalues.

2 Positive-real lemma

Assume that A is Hurwitz stable, (A, B) is controllable, and $D + D^T \succ 0$. The following are equivalent:

1. The system

$$\dot{x} = Ax + Bu, \quad y = Cx + Du, \quad x(0) = 0$$

is passive, *i.e.*, satisfies

$$\int_0^T u(t)^T y(t)\, dt \geq 0$$

for all u and $T \geq 0$.

2. The transfer matrix $H(s) = C(sI - A)^{-1} B + D$ is positive-real, *i.e.*,

$$H(s) + H(s)^* \succeq 0$$

for all s with $\mathrm{Re}(s) \geq 0$.

3. The LMI

$$\begin{bmatrix} A^T P + PA & PB - C^T \\ B^T P - C & -(D + D^T) \end{bmatrix} \preceq 0$$

in the variable $P = P^T$ is feasible. This corresponds to existence of a quadratic storage function $V(x) = x^T P x$ that satisfies

$$\dot{V} - 2u^T y \leq 0.$$

4. There exists a real matrix $P = P^T$ satsifying the ARE

$$A^T P + PA + (PB - C^T)(D + D^T)^{-1}(PB - C^T)^T = 0.$$

5. The sizes of the Jordan blocks corresponding to the pure imaginary eigenvalues of the Hamiltonian matrix

$$M = \begin{bmatrix} A - B(D + D^T)^{-1}C & B(D + D^T)^{-1}B^T \\ -C^T(D + D^T)^{-1}C & -A^T + C^T(D + D^T)^{-1}B^T \end{bmatrix}$$

are all even.

References

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[Yak62] Vladimir A. Yakubovich. The solution of certain matrix inequalities in automatic control theory. *Doklady Akademii Nauk SSSR*, 143(6):1304–1307, 1962.